Weakly monotonic solutions for cooperative games☆

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Abstract

The principle of weak monotonicity for cooperative games states that if a game changes so that the worth of the grand coalition and some player's marginal contribution to all coalitions increase or stay the same, then this player's payoff should not decrease. We investigate the class of values that satisfy efficiency, symmetry, and weak monotonicity. It turns out that this class coincides with the class of egalitarian Shapley values. Thus, weak monotonicity reflects the nature of the egalitarian Shapley values in the same vein as strong monotonicity reflects the nature of the Shapley value. An egalitarian Shapley value redistributes the Shapley payoffs as follows: First, the Shapley payoffs are taxed proportionally at a fixed rate. Second, the total tax revenue is distributed equally among all players.

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1. Introduction

The Shapley value (Shapley, 1953) probably is the most eminent (single-valued) solution concept for cooperative games with transferable utility (TU games). Remarkably, it is not only the unique such concept that satisfies additivity, efficiency, symmetry, and the null player property, but it can be calculated as a player's average marginal contribution to coalitions. Consequently, the Shapley value satisfies a very natural monotonicity condition that conveys desirable incentive properties (Shubik, 1962): whenever a player's marginal contributions weakly increase, his payoff weakly increases. Conversely, Young (1985) shows that this strong monotonicity property (alongside with efficiency and symmetry) is characteristic of the Shapley value. His characterization is also remarkable since it does without additivity, which is a rather technical condition with little economic content.

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Strong monotonicity implies that a player's payoff only depends on his productivity measured by marginal contributions. Hence, the Shapley value reflects individual productivity. Modern societies and institutions, however, distribute their wealth not only based on individual productivity but also on solidarity or egalitarian principles. In order to allow for such principles, strong monotonicity must be waived.

van den Brink et al. (2013) reconcile monotonicity with egalitarianism. In particular, they advocate weak monotonicity as a relaxation of strong monotonicity. Weak monotonicity requires that a player's payoff weakly increases whenever his marginal contributions *and* the grand coalition's worth weakly increase. This principle is particularly attractive in view of the cooperative context and the efficiency assumption. According to efficiency, the grand coalition's worth is to be shared. If this worth does not decrease, there is no need to reduce any player's payoff. Thus, if in addition his individual productivity measured by his marginal contributions to coalitions does not decrease, there is no reason to decrease the player's payoff. van den Brink et al. (2013) show that a solution satisfies efficiency, linearity (or weak covariance), anonymity, and weak monotonicity if and only if it is an egalitarian Shapley value.

The egalitarian Shapley values (Joosten, 1996) are the convex mixtures of the Shapley value and the equal division value. That is, an egalitarian Shapley value redistributes the Shapley payoffs as follows: First, the Shapley payoffs are taxed proportionally at a fixed rate. Second, the total tax revenue is distributed equally among all players.

Our main result states that a value for games with more than two players satisfies efficiency, symmetry, and weak monotonicity if and only if it is an egalitarian Shapley value. Cum grano salis, this is a generalization of Young's characterization of the Shapley value. Moreover, our main result entails that linearity (respectively weak covariance) is redundant in the above characterizations of the egalitarian Shapley values by van den Brink et al. (2013) if there are more than two players.

There are three other generalizations of Young's result in the literature that should be mentioned. Nowak and Radzik (1995) relax the symmetry assumption and give a characterization of the weighted Shapley values (Kalai and Samet, 1987). While their characterization works within the same framework as ours, de Clippel and Serrano (2008) characterize a generalization of the Shapley value for coalitional games with externalities. Hart (2005) provides a characterization of the Maschler-Owen consistent value for non-transferable utility games in a way that generalizes Young's theorem.

Young (1985) also stresses the interest in whether weaker monotonicity criteria are met by other (non-linear) concepts as for example the nucleolus (Schmeidler, 1969) or the core (Gillies, 1959). So far, to the best of our knowledge, no such criteria have been used in order to characterize a popular solution concept or class of solution concepts.

The issue of solidarity is also addressed by Sprumont (1990) who suggests a value that is characterized by Nowak and Radzik (1994) as the "solidarity value". A class of generalizations of this value is given by Casajus and Huettner (2014).

The remainder of this paper is organized as follows. In Section 2, we give basic definitions and notation. In Section 3, we present our main result. Some remarks conclude this paper. An appendix contains the proof of our main result and some complementary findings.

2. Basic definitions and notation

A **(TU) game** is a pair (N, v) consisting of a non-empty and finite set of players N and a **coalition function** $v \in \mathbb{V}(N) := \left\{ f : 2^N \to \mathbb{R} \mid f(\emptyset) = 0 \right\}$. Since we work within a fixed player set, we frequently drop the player set as an argument. In particular, we address $v \in \mathbb{V}$ as a game. Subsets of N are called **coalitions**; $v(S)$ is called the **worth** of coalition S; $v(S \cup \{i\}) - v(S)$ is called the **marginal contribution** of $i \in N$ to $S \subseteq N \setminus \{i\}$. Players $i, j \in N$ are **symmetric** in $v \in \mathbb{V}$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

A **value** on N is a function φ that assigns a payoff vector $\varphi(v) \in \mathbb{R}^N$ to any $v \in \mathbb{V}$. The **Shapley value** (Shapley, 1953) is given by

$$\mathrm{Sh}_i(v) := \sum_{S \subseteq N \setminus \{i\}} \frac{(|N| - |S| - 1)! \cdot |S|!}{|N|!} \cdot (v(S \cup \{i\}) - v(S))$$

for all $v \in \mathbb{V}$ and $i \in N$.

3. Reconciling monotonicity with egalitarianism

Since the Shapley value is an average of marginal contributions, it satisfies a very natural monotonicity condition due to Young (1985).

Strong monotonicity, Mo. For all $v, w \in \mathbb{V}$ and $i \in N$ such that $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$, we have $\varphi_i(v) \geq \varphi_i(w)$.

Strong monotonicity guarantees that a player whose productivity weakly increases does not end up with a lower payoff. Indeed, this property together with efficiency and symmetry, below, is characteristic of the Shapley value.

Efficiency, E. For all $v \in \mathbb{V}$, we have $\sum_{i \in N} \varphi_i(v) = v(N)$.

Symmetry, S. For all $v \in \mathbb{V}$ and $i, j \in N$ such that i and j are symmetric in v, we have $\varphi_i(v) = \varphi_j(v)$.

Efficiency requires that the worth generated by the grand coalition is distributed among the players without losses or gains. Symmetry demands that players who are equally productive obtain the same payoffs.

Theorem 1 (Young, 1985). *The Shapley value is the unique value that satisfies efficiency (**E**), symmetry (**S**), and strong monotonicity (**Mo**).*

Strong monotonicity implies that a player's payoff only depends on his own productivity measured by marginal contributions to coalitions. Therefore, the Shapley value can be viewed as *the* measure of productivity of a player in a TU game.

In contrast, the **equal division value (ED value)** given by

$$\mathrm{ED}_i(v) := \frac{v(N)}{|N|} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N$$

distributes the worth generated by the grand coalition equally among the players. Hence, this value expresses an extremely egalitarian form of solidarity. Clearly, the ED value violates strong monotonicity. However, it satisfies weak monotonicity introduced by van den Brink et al. (2013).

Weak monotonicity, $\mathbf{Mo}^{-}$. For all $v, w \in \mathbb{V}$ and $i \in N$ such that $v(N) \geq w(N)$ and $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$, we have $\varphi_i(v) \geq \varphi_i(w)$.

According to this property, a player's payoff weakly increases whenever his marginal contributions *and* the grand coalition's worth weakly increase. This property weakens strong monotonicity in a plausible way. Whenever the worth generated by the grand coalition weakly increases, no player necessary *has* to loose. If in addition a player's productivity weakly increases, this player's payoff *should* not decrease.

The **egalitarian Shapley values** (Joosten, 1996) Sh^{α}, $\alpha \in [0,1]$ given by

$$\mathrm{Sh}_i^{\alpha}(v) = \alpha \cdot \mathrm{Sh}_i(v) + (1-\alpha) \cdot \mathrm{ED}_i(v) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N \tag{1}$$

also satisfy weak monotonicity. Indeed, van den Brink et al. (2013) use this property in order to single out the egalitarian Shapley values from the class of linear, efficient, and symmetric solutions.[1]

Linearity, L. For all $v, w \in \mathbb{V}$ and $\rho \in \mathbb{R}$, we have $\varphi(v+w) = \varphi(v) + \varphi(w)$ and $\varphi(\rho \cdot v) = \rho \cdot \varphi(v)$.[2]

Theorem 2 (van den Brink et al., 2013). *A value φ satisfies efficiency (**E**), linearity (**L**), symmetry (**S**), and weak monotonicity ($\mathbf{Mo}^{-}$) if and only if there exists an $\alpha \in [0,1]$ such that $\varphi = \mathrm{Sh}^{\alpha}$.*

Inspired by Young (1985), we now investigate the class of values that satisfy efficiency, symmetry, and weak monotonicity. This is of interest for at least two reasons. First, it is difficult to motivate linearity in economic terms. Second, Young's characterization of the Shapley value shows that strong monotonicity essentially reflects the nature of the Shapley value. Cum grano salis, it turns out that weak monotonicity reflects the nature of the egalitarian Shapley values in the same vein.

Theorem 3. *For $|N| \neq 2$, a value φ satisfies efficiency (**E**), symmetry (**S**), and weak monotonicity ($\mathbf{Mo}^{-}$) if and only if there exists an $\alpha \in [0,1]$ such that $\varphi = \mathrm{Sh}^{\alpha}$.*

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[1] Actually, they employ anonymity instead of symmetry. Yet, Malawski (2008, Theorem 2) shows that efficiency, linearity, and symmetry already entail anonymity: For all $v \in \mathbb{V}$, $i \in N$, and all bijections $\pi : N \to N$, we have $\varphi_{\pi(i)}(v \circ \pi^{-1}) = \varphi_i(v)$, where $v \circ \pi^{-1} \in \mathbb{V}$ is given by $v \circ \pi^{-1}(S) = v(\pi^{-1}(S))$, $S \subseteq N$.

[2] For all $v, w \in \mathbb{V}$ and $\rho \in \mathbb{R}$, the coalition functions $v + w \in \mathbb{V}$ and $\rho \cdot v \in \mathbb{V}$ are given by $(v+w)(S) = v(S) + w(S)$ and $(\rho \cdot v)(S) = \rho \cdot v(S)$ for all $S \subseteq N$.

The proof of Theorem 3 can be found in Appendix A. Appendix B contains the counterexample to our characterization for $|N| = 2$. The non-redundancy of our characterization for $|N| > 2$ is indicated in Appendix C.

Note that Theorem 3 implies Theorem 1 for $|N| \neq 2$ since Sh^α satisfies strong monotonicity only if $\alpha = 1$. Moreover, Theorem 3 entails that linearity is redundant in the characterization of the egalitarian Shapley values by Theorem 2 for $|N| > 2$.[3] The example presented in Appendix B establishes that linearity is not implied by efficiency, symmetry, and weak monotonicity if $|N| = 2$.

van den Brink et al. (2013, Theorem 4.5) provide another characterization of the egalitarian Shapley values for $|N| > 2$ using efficiency, anonymity, weak monotonicity, and weak covariance, below.

Weak covariance. For all $v \in \mathbb{V}$, $i \in N$, $j, k \in N \backslash \{i\}$, and $\rho \in \mathbb{R}$, we have $\varphi_j\left(v + \rho \cdot u_{\{i\}}\right) - \varphi_j(v) = \varphi_k\left(v + \rho \cdot u_{\{i\}}\right) - \varphi_k(v)$, where $u_{\{i\}} \in \mathbb{V}$ is given by $u_{\{i\}}(S) = 1$ if $i \in S$ and $u_{\{i\}}(S) = 0$ if $i \in N \setminus S$ for all $S \subseteq N$.

As a relaxation of the fairness property[4] due to van den Brink (2001), weak covariance might have more economic appeal than linearity. Weak covariance states that the payoffs of two players j and k are changed by the same amount if a third player i's marginal contributions to coalitions are changed by a fixed amount ρ. This also is the case if j and k are not symmetric. Then, weak covariance directly requires unequals to be affected equally. In this sense, weak covariance is not as innocuous as it might seem at first glance. Also note that linearity neither implies nor is implied by weak covariance. In view of Theorem 3, however, weak covariance is redundant in the characterization of the egalitarian Shapley values by van den Brink et al. (2013, Theorem 4.5).[5]

The hypothesis of weak monotonicity consists of two conditions. Separating these conditions results in the notions of strong monotonicity and grand coalition monotonicity, below.

Grand coalition monotonicity, GMo. For all $v, w \in \mathbb{V}$ and $i \in N$ such that $v(N) \geq w(N)$, we have $\varphi_i(v) \geq \varphi_i(w)$.

Grand coalition monotonicity requires that a player's payoff weakly increases whenever the worth of the grand coalition weakly increases. It is clear that the ED-value is the only egalitarian Shapley value that satisfies grand coalition monotonicity. van den Brink (2007, Theorem 3.3) uses a weaker property—coalitional monotonicity[6]—in order to characterize the ED-value. This yields the following corollary.

Corollary 4. *The ED-value is the unique value that satisfies efficiency (**E**), symmetry (**S**), and grand coalition monotonicity (**GMo**).*

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[3]The example given by van den Brink et al. (2013) in order to show the logical independence of linearity from the other axioms does not satisfy weak monotonicity.

[4]Fairness: For all $v, w \in \mathbb{V}$ and $i, j \in N$ such that i and j are symmetric in w, we have $\varphi_i(v + w) - \varphi_i(v) = \varphi_j(v + w) - \varphi_j(v)$.

[5]The example given by van den Brink et al. (2013) in order to show the logical independence of weak covariance from the other axioms does not satisfy weak monotonicity.

[6]Coalitional monotonicity: For all $v, w \in \mathbb{V}$ and $i \in N$ such that $v(S) \geq w(S)$ for all $S \subseteq N$, $S \ni i$, we have $\varphi_i(v) \geq \varphi_i(w)$.

4. Concluding remarks

In this paper, we suggest a new characterization of the class of egalitarian Shapley values using efficiency, symmetry, and weak monotonicity. This characterization corresponds to the characterization of the egalitarian Shapley values given by Casajus and Huettner (2013) as Young's (1985) characterization of the Shapley value corresponds to the original characterization given by Shapley (1953) himself.

The monotonicity property of the Shapley value is often used as an argument to support the application of the Shapley value. Moreover, characterizations of related solution concepts essentially rely on Young's theorem (e.g. Maniquet, 2003; Bergantiños and Vidal-Puga, 2007). Therefore, it might be interesting to study the consequences of a relaxation of the property derived from strong monotonicity in such applications.

Appendix A. Proof of Theorem 3

First, we provide some further definitions and notation. For any set of players N, let $n := |N|$. For $v \in \mathbb{V}$, let the equivalence relation $\sim_v$ on N be given as follows. For all $i, j \in N$, we have $i \sim_v j$ if i and j are symmetric in v. For $v \in \mathbb{V}$, let $\mathcal{N}(v)$ denote the partition of N induced by $\sim_v$. Set

$$\#v := \max_{C \in \mathcal{N}(v)} |C|. \tag{A.1}$$

A game $v \in \mathbb{V}$ is called symmetric if $\#v = n$. Let $\mathbb{V}^* = \{v \in \mathbb{V} \,|\#v = n\}$ denote the set of symmetric games. Set $\mathbb{R}^N := \{f : N \to \mathbb{R}\}$ and $i \mapsto x_i := x(i)$ for all $i \in N$. For $x \in \mathbb{R}^N$, the modular game $m_x \in \mathbb{V}$ is given by $m_x(S) = \sum_{i \in S} x_i$ for all $S \subseteq N$. Note that $m_x = u_{\{i\}}$ if $x_i = 1$ and $x_j = 0$ for all $j \in N \setminus \{i\}$. The null game $\mathbf{0} \in \mathbb{V}^*$ is given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$.

By van den Brink et al. (2013, Theorem 4.3), Sh^α, $\alpha \in [0, 1]$ obeys **E**, **S**, and **Mo**$^-$. Let the value φ satisfy **E**, **S**, and **Mo**$^-$. By **E**, $\varphi = \mathrm{Sh}^\alpha$ for all $\alpha \in [0, 1]$, if $n = 1$. Let now $n > 2$. The remainder of the proof consists of two parts. First, we establish with a series of claims that there is some $\alpha \in [0, 1]$ such that for all symmetric games $v \in \mathbb{V}^*$, $i \in N$, $j \in N \setminus \{i\}$, and $x \in \mathbb{R}^N$ such that $x_j = 0$, we have $\varphi_i(v + m_x) - \varphi_j(v + m_x) = \alpha \cdot x_i$. Second, we use this insight in order to show $\varphi(v) = \mathrm{Sh}^\alpha(v)$ if $\#v = n - 1$ and establish $\varphi(v) = \mathrm{Sh}^\alpha(v)$ by induction on $\#v$.

Claim 1, C1. For $v \in \mathbb{V}^*$, there exists a mapping $F_i^v : \mathbb{R}^2 \to \mathbb{R}$ for each $i \in N$ such that

$$\varphi_i(v + m_x) = F_i^v\left(x_i, \sum_{\ell \in N} x_\ell\right) \qquad \text{for all } x \in \mathbb{R}^N. \tag{A.2}$$

Mo$^-$ implies $\varphi_i(v + m_x) = \varphi_i(v + m_y)$ for $i \in N$ and $x, y \in \mathbb{R}^N$ whenever $x_i = y_i$ and $\sum_{i \in N} x_i = \sum_{i \in N} y_i$. This is tantamount to **C1**.

Claim 2, C2. F_i^v does not depend on the choice of $i \in N$, i.e., for all $v \in \mathbb{V}^*$ and $i, j \in N$, we have $F_i^v = F_j^v =: F^v$.

Let $v \in \mathbb{V}^*$, $a, c \in \mathbb{R}$, $x, y \in \mathbb{R}^N$, $i \in N$, and $j \in N \setminus \{i\}$ be such that

$$x_i = a \qquad \text{and} \qquad x_k = \frac{c-a}{n-1} \quad \text{for all } k \in N \setminus \{i\} \tag{A.3}$$

and

$$y_j = a \qquad \text{and} \qquad y_k = \frac{c-a}{n-1} \quad \text{for all } k \in N \setminus \{j\}\,. \tag{A.4}$$

Hence, we have

$$L := \varphi_\ell\left(v + m_x\right) \overset{\text{(A.3),(A.4),}\mathbf{Mo}^-}{=} \varphi_\ell\left(v + m_y\right) \qquad \text{for all } \ell \in N \setminus \{i, j\} \tag{A.5}$$

$$\varphi_j\left(v + m_x\right) \overset{\text{(A.3),}\mathbf{S}}{=} L \tag{A.6}$$

$$\varphi_i\left(v + m_y\right) \overset{\text{(A.4),}\mathbf{S}}{=} L \tag{A.7}$$

and therefore

$$\begin{aligned} F_i^v(a,c) &\overset{\text{(A.2),(A.3)}}{=} \varphi_i\left(v + m_x\right) \\ &\overset{\mathbf{E}\text{,(A.5),(A.6),(A.7)}}{=} \left(v + m_x\right)(N) - (n-1) \cdot L \\ &\overset{\text{(A.3),(A.4)}}{=} \left(v + m_y\right)(N) - (n-1) \cdot L \\ &\overset{\mathbf{E}\text{,(A.5),(A.6),(A.7)}}{=} \varphi_j\left(v + m_y\right) \\ &\overset{\text{(A.2),(A.4)}}{=} F_j^v(a,c)\,, \end{aligned}$$

which proves **C2**.

For $c \in \mathbb{R}$ and $v \in \mathbb{V}^*$, the mapping $\Phi^{v,c} : \mathbb{R} \to \mathbb{R}$ is defined by

$$\Phi^{v,c}(a) := F^v(a,c) - F^v(0,c) \qquad \text{for all } a \in \mathbb{R}. \tag{A.8}$$

Claim 3, C3. For all $a, c, \rho \in \mathbb{R}$, we have $\Phi^{v,c}(\rho \cdot a) = \rho \cdot \Phi^{v,c}(a)$.

By $\mathbf{Mo}^-$, the mappings F^v have the following property. For all $a, a', c, c' \in \mathbb{R}$ such that $a \geq a'$ and $c \geq c'$, we have

$$F^v(a,c) \geq F^v(a',c')\,. \tag{A.9}$$

Fix $i, j \in N$, $i \neq j$. For $a, b, c \in \mathbb{R}$, let $x, y \in \mathbb{R}^N$ be given by $x_i = a$, $x_j = b$, $y_i = a + b$, $y_j = 0$, and $x_k = y_k = \frac{c-a-b}{n-2}$ for all $k \in N \setminus \{i, j\}$. For all $v \in \mathbb{V}^*$, we have

$$\begin{aligned} &F^v(a,c) + F^v(b,c) + (n-2) \cdot F^v\left(\frac{c-a-b}{n-2}, c\right) \\ &\overset{\text{(A.2),}\mathbf{C2}}{=} \sum_{\ell \in N} \varphi_\ell\left(v + m_x\right) \\ &\overset{\mathbf{E}}{=} \sum_{\ell \in N} \varphi_\ell\left(v + m_y\right) \\ &\overset{\text{(A.2),}\mathbf{C2}}{=} F^v(a+b,c) + F^v(0,c) + (n-2) \cdot F^v\left(\frac{c-a-b}{n-2}, c\right). \end{aligned} \tag{A.10}$$

By (A.8) and (A.10), we obtain

$$\Phi^{v,c}(a) + \Phi^{v,c}(b) = \Phi^{v,c}(a+b) \qquad \text{for all } a, b, c \in \mathbb{R}. \tag{A.11}$$

This already entails that $\Phi^{v,c}(\rho \cdot a) = \rho \cdot \Phi^{v,c}(a)$ for all $a \in \mathbb{R}$, $\rho \in \mathbb{Q}$. By (A.9) and (A.8), the mapping $\Phi^{v,c}$ is monotonic, i.e., we have $\Phi^{v,c}(a) \geq \Phi^{v,c}(b)$ for all $a, b \in \mathbb{R}$ such that $a \geq b$. Since $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, (A.11) entails **C3**.

Claim 4, C4. $\Phi^{v,c}(1)$ does not depend on the choice of c, i.e., $\Phi^{v,c}(1) = \Phi^{v,c'}(1) =: \Phi^{v}(1)$ for all $c, c' \in \mathbb{R}$.

Let $c, c' \in \mathbb{R}$, $c > c'$. Suppose $\Phi^{v,c}(1) \neq \Phi^{v,c'}(1)$. Then, we have

$$\begin{aligned} F^{v}(a,c) - F^{v}(a,c') &\overset{(A.8)}{=} \Phi^{v,c}(a) - \Phi^{v,c'}(a) + F^{v}(0,c) - F^{v}(0,c') \\ &\overset{\mathbf{C3}}{=} a \cdot \left(\Phi^{v,c}(1) - \Phi^{v,c'}(1)\right) + F^{v}(0,c) - F^{v}(0,c') \end{aligned}$$

for all $a \in \mathbb{R}$. Hence, one can find some $a \in \mathbb{R}$ such that $F^{v}(a,c) < F^{v}(a,c')$, contradicting (A.9).

Claim 5, C5. $\Phi^{v}(1)$ does not depend on the choice of $v \in \mathbb{V}^{*}$, i.e., $\Phi^{v}(1) = \Phi^{w}(1) =: \Phi(1)$ for all $v, w \in \mathbb{V}^{*}$.

Let $v, w \in \mathbb{V}^{*}$ and $i, j \in N$, $i \neq j$. One can choose $z \in \mathbb{V}^{*}$ such that $z(N) \geq v(N)$, $z(N) \geq w(N)$, and $z(S \cup \{i\}) - z(S) \geq v(S \cup \{i\}) - v(S)$ and $z(S \cup \{i\}) - z(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$. Suppose $\Phi^{z}(1) \neq \Phi^{v}(1)$. Then, we have

$$\begin{aligned} &\varphi_i\left(z + \rho \cdot u_{\{i\}} + (1-\rho) \cdot u_{\{j\}}\right) - \varphi_i\left(v + \rho \cdot u_{\{i\}} + (1-\rho) \cdot u_{\{j\}}\right) - \left(F^{z}(0,1) - F^{v}(0,1)\right) \\ &\overset{(A.2),\mathbf{C2},(A.8),\mathbf{C3},\mathbf{C4}}{=} \rho \cdot \left(\Phi^{z}(1) - \Phi^{v}(1)\right) \qquad \text{for all } \rho \in \mathbb{R}. \end{aligned}$$

Hence, one can find some $\rho \in \mathbb{R}$ such that $\varphi_i\left(z + \rho \cdot u_{\{i\}}\right) < \varphi_i\left(v + \rho \cdot u_{\{i\}}\right)$, contradicting $\mathbf{Mo}^{-}$. Thus, $\Phi^{z}(1) = \Phi^{v}(1)$. Analogously, one shows $\Phi^{z}(1) = \Phi^{w}(1)$. This establishes **C5**.

Set

$$\alpha := \Phi(1). \tag{A.12}$$

Claim 6, C6. $\alpha \in [0,1]$.

We have

$$\alpha \overset{(A.12)}{=} \Phi(1) \overset{\mathbf{C5},\mathbf{C4}}{=} \Phi^{\mathbf{0},1}(1) \overset{\mathbf{C2},(A.8)}{=} F^{\mathbf{0}}(1,1) - F^{\mathbf{0}}(0,1). \tag{A.13}$$

By (A.13) and (A.9), $\alpha \geq 0$. For all $i, j \in N$, $j \neq i$, we have

$$\varphi_i\left(u_{\{i\}}\right) - \varphi_j\left(u_{\{i\}}\right) \overset{(A.2),\mathbf{C2}}{=} F^{\mathbf{0}}(1,1) - F^{\mathbf{0}}(0,1) \overset{(A.13)}{=} \alpha. \tag{A.14}$$

Since $\varphi_\ell\left(u_{\{i\}}\right) \overset{\mathbf{Mo}^{-}}{\geq} \varphi_\ell(\mathbf{0}) \overset{\mathbf{E},\mathbf{S}}{=} 0$ for $\ell \in N$, we obtain

$$\varphi_i\left(u_{\{i\}}\right) - \varphi_j\left(u_{\{i\}}\right) \overset{\mathbf{E}}{=} 1 - \sum_{\ell \in N \setminus \{i\}} \varphi_\ell\left(u_{\{i\}}\right) - \varphi_j\left(u_{\{i\}}\right) \leq 1,$$

i.e., $\alpha \leq 1$.

Claim 7, C7. For all $v \in \mathbb{V}^*$, $i \in N$, $j \in N \setminus \{i\}$, and $x \in \mathbb{R}^N$ such that $x_j = 0$, we have $\varphi_i (v + m_x) - \varphi_j (v + m_x) = \alpha \cdot x_i$.

With v, i, j, and x as in the claim, set $X := \sum_{\ell \in N} x_\ell$. Then, we have

$$\begin{aligned} & \varphi_i (v + m_x) - \varphi_j (v + m_x) \\ \overset{(A.2),\mathbf{C2},(A.8)}{=} & \Phi^{v,X} (x_i) + F^v (0, X) - \left[\Phi^{v,X} (0) + F^v (0, X)\right] \\ \overset{\mathbf{C3},\mathbf{C4},\mathbf{C5},\mathbf{C6}}{=} & \alpha \cdot x_i, \end{aligned}$$

which establishes **C7**.

Fix $\alpha \in [0, 1]$. It remains to show that there is at most one value φ that satisfies **E**, **S**, **Mo**$^-$, and $\varphi_i \left(u_{\{i\}}\right) - \varphi_j \left(u_{\{i\}}\right) = \alpha$ for $i \in N$ and $j \in N \setminus \{i\}$. Let φ be such a value. We show $\varphi (v) = \mathrm{Sh}^\alpha (v)$ by induction on $\#v$ defined in (A.1).

Induction basis. We show that $\varphi (v) = \mathrm{Sh}^\alpha (v)$ for all $v \in \mathbb{V}$ such that $\#v \in \{n, n - 1\}$.

For $\#v = n$, this is immediate from (1), **E**, and **S**. Let now $v \in \mathbb{V}$ be such that $\#v = n-1$, i.e., $\mathcal{N} (v) = \{N \setminus \{i\}, \{i\}\}$ for some $i \in N$. By Pintér (2012, Definition 3.2, Proposition 3.6), there is some $w \in \mathbb{V}^*$ such that $v (S \cup \{i\}) - v (S) = w (S \cup \{i\}) - w (S)$ for $S \subseteq N \setminus \{i\}$. Fix $k \in N \setminus \{i\}$. Let $z \in \mathbb{V}$ be given by

$$z := w + (v (N) - w (N)) \cdot u_{\{k\}}. \tag{A.15}$$

This implies $z (N) = v (N)$ and $v (S \cup \{i\}) - v (S) = z (S \cup \{i\}) - z (S)$ for $S \subseteq N \setminus \{i\}$. Hence, we have

$$\varphi_i (v) \overset{\mathbf{Mo}^-}{=} \varphi_i (z) \qquad \text{and} \qquad \mathrm{Sh}^\alpha_i (v) \overset{\mathbf{Mo}^-}{=} \mathrm{Sh}^\alpha_i (z). \tag{A.16}$$

By $w \in \mathbb{V}^*$, (A.15), and **C7**, we obtain

$$\varphi_k (z) - \varphi_i (z) = \alpha \cdot (v (N) - w (N)). \tag{A.17}$$

By (A.15) and **S**, we get $\varphi_i (z) = \varphi_\ell (z)$ for all $\ell \in N \setminus \{k\}$. Hence,

$$\varphi_k (z) + (n - 1) \cdot \varphi_i (z) \overset{\mathbf{E}}{=} z (N) \overset{(A.15)}{=} v (N). \tag{A.18}$$

Solving (A.17) and (A.18) for $\varphi_i (z)$ yields

$$\varphi_i (z) = \frac{w (N)}{n} + (1 - \alpha) \cdot \frac{v (N) - w (N)}{n} \overset{(1)}{=} \mathrm{Sh}^\alpha_i (z). \tag{A.19}$$

By (A.16) and (A.19), $\varphi_i (v) = \mathrm{Sh}^\alpha_i (v)$. Since any two players in $N \setminus \{i\}$ are symmetric in v and by **E** and **S**, we also have $\varphi_\ell (v) = \mathrm{Sh}^\alpha_\ell (v)$ for $\ell \in N \setminus \{i\}$.

Induction hypothesis, IH. Suppose $\varphi (w) = \mathrm{Sh}^\alpha (w)$ for all $w \in \mathbb{V}$ such that $\#w \geq t$, $t < n$.

Induction step. We show that $\varphi(v) = \mathrm{Sh}^{\alpha}(v)$ for all $v \in \mathbb{V}$ such that $\#v = t-1$.

Let $v \in \mathbb{V}$ be such that $\#v = t-1$. Note that $\#v \leq n-2$. Fix $C \in \mathcal{N}(v)$ such that $|C| = \#v$. Fix $i, j \in N \setminus C$, $i \neq j$. By Pintér (2012, Definition 3.2, Proposition 3.6), there is some $w \in \mathbb{V}$ such that $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ for $S \subseteq N \setminus \{i\}$ and such that any two players in $C \cup \{i\}$ are symmetric in w, i.e., $\#w \geq t$. Let $z \in \mathbb{V}$ be given by $z := w + (v(N) - w(N)) \cdot u_{\{j\}}$. This implies $\#z \geq \#w$, $z(N) = v(N)$, and $v(S \cup \{i\}) - v(S) = z(S \cup \{i\}) - z(S)$ for $S \subseteq N \setminus \{i\}$. Hence, we have

$$\varphi_i(v) \overset{\mathbf{Mo}^-}{=} \varphi_i(z) \overset{\mathbf{IH}}{=} \mathrm{Sh}_i^{\alpha}(z) \overset{\mathbf{Mo}^-}{=} \mathrm{Sh}_i^{\alpha}(v).$$

Since $i \in N \setminus C$ was chosen arbitrarily, we have $\varphi_\ell(v) = \mathrm{Sh}_\ell^{\alpha}(v)$ for $\ell \in N \setminus C$. Since any two players in C are symmetric in v and by **E** and **S**, this entails $\varphi_\ell(v) = \mathrm{Sh}_\ell^{\alpha}(v)$ for $\ell \in C$. $\square$

Appendix B. Counterexample to Theorem 3 for $n = 2$

Theorem 3 fails for $n = 2$. Consider the redistribution rule $\varphi^{\heartsuit}$ on $N = \{1, 2\}$ given by

$$\left(\varphi_1^{\heartsuit}(v), \varphi_2^{\heartsuit}(v)\right) = \begin{cases} (\mathrm{Sh}_1(v), \mathrm{Sh}_2(v)), & \mathrm{Sh}_1(v) \geq 0 \quad \text{and} \quad \mathrm{Sh}_2(v) \geq 0, \\ (0, v(N)), & \mathrm{Sh}_1(v) < 0 \quad \text{and} \quad \mathrm{Sh}_2(v) > 0 \wedge v(N) \geq 0, \\ (v(N), 0), & \mathrm{Sh}_1(v) < 0 \quad \text{and} \quad \mathrm{Sh}_2(v) > 0 \wedge v(N) < 0, \\ (\mathrm{Sh}_1(v), \mathrm{Sh}_2(v)), & \mathrm{Sh}_1(v) \leq 0 \quad \text{and} \quad \mathrm{Sh}_2(v) \leq 0, \\ (0, v(N)), & \mathrm{Sh}_1(v) > 0 \quad \text{and} \quad \mathrm{Sh}_2(v) < 0 \wedge 0 \geq v(N), \\ (v(N), 0), & \mathrm{Sh}_1(v) > 0 \quad \text{and} \quad \mathrm{Sh}_2(v) < 0 \wedge v(N) > 0 \end{cases}$$

for all $v \in \mathbb{V}(N)$. It is straightforward to show that $\varphi^{\heartsuit}$ meets **E**, **S**, and $\mathbf{Mo}^-$.

Appendix C. Non-redundancy of Theorem 3 for $n > 2$

Our characterization is non-redundant for $n > 2$. The value $\varphi^{\mathbf{E}}$ given by $\varphi_i^{\mathbf{E}}(v) = 0$ for all $v \in \mathbb{V}$ and $i \in N$ satisfies **S** and $\mathbf{Mo}^-$ but not **E**. Fix $\mathring{i} \in N$. The value $\varphi^{\mathbf{S}}$ given by $\varphi_{\mathring{i}}^{\mathbf{S}}(v) = v(N)$ and $\varphi_i^{\mathbf{S}}(v) = 0$ for all $v \in \mathbb{V}$ and $i \in N \setminus \{\mathring{i}\}$ satisfies **E** and $\mathbf{Mo}^-$ but not **S**. The value $\varphi^{\mathbf{Mo}^-}$ given by $\varphi^{\mathbf{Mo}^-}(v) = 2 \cdot \mathrm{Sh}(v) - \mathrm{ED}(v)$ for all $v \in \mathbb{V}$ satisfies **E** and **S** but not $\mathbf{Mo}^-$.

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